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Mh4718 Week 1

Week 1

0.1 Representation of Numbers and precision

0.1.1 Representation

Integers are usually represented using a so-called *place value* notation. A set of, say b basic symbols is chosen,(b being an integer ≥ 2) and a unique representation of n is obtained by finding integers $d_0, d_1, d_2, \ldots, d_n$ such that

$$n = d_n \times b^n + d_{n-1} \times b^{n-1} + \cdots + d_1 \times b + d_0$$

and then n is denoted by

$$d_n d_{n-1} \ldots d_1 d_0$$

or by

$$(d_n d_{n-1} \ldots d_1 d_0)_b$$

if there is any ambiguity about the base.

The use of ten as a number base for representation is clearly related to our anatomy. Other number bases are more appropriate to other circumstances. Computers and calculators use base two (binary)representation of numbers which requires only two symbols 0 and 1 corresponding with two possible states (on and off) of a transistor.
Thus, for example, if 1100101 is a base two representation then:
$1100101 =$
$1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
$= 64 + 32 + 0 + 0 + 4 + 0 + 1$
$= 101$

1011 could be a base ten or a base two number. Sometimes the context will make it clear which base we are working in but in case of possible confusion we will write, say, $(1011)_2$ to indicate a base two representation and $(1011)_{\text{ten}}$ to indicate a base ten representation.

To convert an integer to base two representation we rebundle in twos rather than in tens.

Example 0.1

Express 155 in base two place value notation:

$$155 \div 2 = 77 + \text{ remainder } 1.$$

That is, $155 = 77$ bundles of two $+ 1$. Therefore we will have 1 in the units place.

$$77 \div 2 = 38 + \text{ remainder } 1.$$

That is, $77 = 38$ bundles of two $+ 1$
$\Rightarrow 155 = 38$ bundles of 2^2 $+ 1$ bundle of 2+1.
Therefore there will be 1 in the 2^1 column.

$$38 \div 2 = 19 + \text{ remainder } 0.$$

That is, $38 = 19$ bundles of two + remainder 0
$\Rightarrow 155 = 19$ bundles of 2^3 $+ 0$ bundles of 2^2 $+ 1$ bundle of 2 +1.
Therefore there will be 0 in the 2^2 column.

$$19 \div 2 = 9 + \text{ remainder } 1.$$

That is, $19 = 9$ bundles of two + remainder 1
$\Rightarrow 155 = 9$ bundles of 2^4 +1 bundle of 2^3 $+ 0$ bundles of 2^2 $+ 1$ bundle of 2+1.
Therefore there will be 1 in the 2^3 column.

$$9 \div 2 = 4 + \text{ remainder } 1.$$

That is, $9 = 4$ bundles of two + remainder 1
$\Rightarrow 155 = 4$ bundles of 2^5 +1 bundle of 2^4 +1 bundle of 2^3 $+ 0$ bundles of 2^2 $+$ 1 bundle of 2+1.
Therefore there will be 1 in the 2^4 column.

$$4 \div 2 = 4 + \text{remainder } 0.$$

That is, $4 = 2$ bundles of two $+$ remainder $0 \Rightarrow$
155 =2 bundles of 2^6+ 0 bundles of 2^5 +1 bundle of 2^4 +1 bundle of 2^3 + 0

bundles of 2^2 + 1 bundle of 2+1.
Therefore there will be 0 in the 2^5 column.

$$2 \div 2 = 1 + \text{ remainder } 0.$$

That is, 2 = 1 bundles of two + remainder 0
$\Rightarrow$ 155 =1 bundle of 2^7+0 bundles of 2^6+ 0 bundles of 2^5 +1 bundle of 2^4 +1 bundle of 2^3 + 0 bundles of 2^2 + 1 bundle of 2+1.
Therefore there will be 0 in the 2^6 column and 1 in the 2^7 column.
Therefore 155 = $(10011011)_2$.

This process can be streamlined as follows:

2	155	
2	77	R 1
2	38	R 1
2	19	R 0
2	9	R 1
2	4	R 1
2	2	R 0
2	1	R 0
	0	R 1

We get the binary representation of 155 by reading the remainders from bottom to top.

Using negative powers of the base we can also represent real numbers which are not integers using place value notation but, in this case, the representation may not be finite in length.

If we seek to extend the base ten place value notation scheme to, say, $\frac{1}{8}$ we proceed as follows:

Since $\frac{1}{8} < 1$ we cannot break it into units and bundles of ten. Instead we first seek to break it into bundles of $\frac{1}{10}$'s, that is, bundles of 10^{-1}. We ask then how many $\frac{1}{10}$'s are in $\frac{1}{8}$.

$$\frac{1}{8} \div \frac{1}{10} = \frac{1}{8} \times 10 = \frac{10}{8} = 1\frac{2}{8}$$

$$\Rightarrow \frac{1}{8} = (1 + \frac{2}{8}) \times \frac{1}{10} = 1 \times \frac{1}{10} + \frac{2}{8} \times \frac{1}{10} = 1 \times 10^{-1} + \frac{2}{80}$$

We now seek to break the left over $\frac{2}{80}$ into bundles of $\frac{1}{100}$'s.
We ask how many $\frac{1}{100}$'s are in $\frac{2}{80}$:

$$\frac{2}{80} \div \frac{1}{100} = \frac{2}{80} \times 100 = \frac{200}{80} = \frac{20}{8} = 2\frac{4}{8}$$

$$\Rightarrow \frac{2}{80} = (2 + \frac{4}{8}) \times \frac{1}{100} = 2 \times \frac{1}{100} + \frac{4}{8} \times \frac{1}{100} = 2 \times 10^{-2} + \frac{4}{800}$$

And so we have:

$$\frac{1}{8} = 1 \times \frac{1}{10} + 2 \times 10^{-2} + \frac{4}{800}$$

We then seek to break the left over $\frac{4}{800}$ into bundles of $\frac{1}{1000}$'s.

$$\frac{4}{800} \div \frac{1}{1000} = \frac{4}{800} \times 1000 = \frac{4000}{800} = \frac{40}{8} = 5$$

$$\Rightarrow \frac{4}{800} = 5 \times \frac{1}{1000}$$

And so get

$$\frac{1}{8} = 1 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3}$$

.
And we have succeed in expressing $\frac{1}{8}$ as a sum of powers of 10. And we write

$$\frac{1}{8} = 0.125$$

The process we used above to do this is tedious but note the the digits in the representation 0.125 were obtained by the steps

$$\frac{10}{8} = \underline{1}\,\frac{2}{8}$$

$$\frac{20}{8} = \underline{2}\,\frac{4}{8}$$

$$\frac{40}{8} = \underline{5}$$

And this sequence corresponds with the usual method of dividing 8 into 1:

$$8 \,\big|\, \underline{1.{}_{1}0{}_{2}0{}_{4}0}$$
$$0.\;1\;2\;5$$

This technique is quicker and we use it for actual calculations.

However, if we attempt to represent, say, $\frac{1}{3}$ by this means we run into a never ending process:
How many $\frac{1}{10}$'s in $\frac{1}{3}$?:

$$\frac{1}{3} \div \frac{1}{10} = \frac{1}{3} \times 10 = \frac{10}{3} = 3\frac{1}{3}$$

$$\Rightarrow \frac{1}{3} = (1 + \frac{1}{3}) \times \frac{1}{10} = 3 \times 10^{-1} + \frac{1}{30}$$

How many $\frac{1}{100}$'s in $\frac{1}{30}$?:

$$\frac{1}{30} \div \frac{1}{100} = \frac{1}{30} \times 100 = \frac{10}{3} = 3\frac{1}{3}$$

$$\Rightarrow \frac{1}{30} = (3 + \frac{1}{3}) \times \frac{1}{100} = 3 \times 10^{-2} + \frac{1}{300}$$

$$\Rightarrow \frac{1}{3} = 3 \times 10^{-1} + 3 \times 10^{-2} + \frac{1}{300}$$

How many $\frac{1}{1000}$'s in $\frac{1}{300}$?:

$$\frac{1}{300} \div \frac{1}{1000} = \frac{1}{300} \times 1000 = \frac{10}{3} = 3\frac{1}{3}$$

$$\Rightarrow \frac{1}{300} = (3 + \frac{1}{3}) \times \frac{1}{1000} = 3 \times 10^{-3} + \frac{1}{3000}$$

$$\Rightarrow \frac{1}{3} = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{3} + \frac{1}{3000}$$

Clearly this process will never terminate. After n steps we will arrive at:

$$\frac{1}{3} = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{3} + \cdots + 3 \times 10^{-n} + \frac{1}{3 \times 10^{n}}$$

That is,

$$\frac{1}{3} = \underbrace{0.3333\ldots\ldots3}_{n\ decimalplaces} + \frac{1}{3 \times 10^{n}}$$

We note, however, that $3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{3} + \cdots + 3 \times 10^{-n}$ becomes arbitrarily close to $\frac{1}{3}$ - the 'leftover' $\frac{1}{3 \times 10^{n}}$ is becoming arbitrarily small as n gets larger.

We say then that the decimal $\underbrace{0.3333\ldots\ldots3}_{n\ decimalplaces}$ 'converges' to $\frac{1}{3}$ as n goes to ∞.

This is usually expressed by saying that that the decimal representation of $\frac{1}{3}$ is infinite with

$$\frac{1}{3} = 0.33333333\ldots\ldots$$

or more properly

$$\frac{1}{3} = 0.\dot{3}$$

To convert a non-integer rational to binary we proceed exactly as in base ten except that we multipy by two rather than by ten. This is because we want to break the number up into bundles of $\frac{1}{2}$, $\frac{1}{2^2}$ etc. That is, we want to successively divide by $\frac{1}{2}$ which is the same as multiplying by 2.

Example 0.2

Write 0.375 in binary representation:

$$\begin{array}{r} 0.375 \\ \underline{\times 2} \\ \boxed{0}.750 \\ \underline{\times 2} \\ \boxed{1}.50 \\ \underline{\times 2} \\ \boxed{1}.0 \end{array}$$

So we see that there are 0 $\frac{1}{2}$'s, 1 $\frac{1}{2^2}$'s and 1 $\frac{1}{2^3}$'s which means that the binary representation of 0.375 is 0.011.

Example 0.3

Write $\frac{3}{8}$ in binary place value representation:

$$\frac{3}{8} \times 2 = \frac{6}{8} = \boxed{0}\frac{6}{8}$$

$$\frac{6}{8} \times 2 = \frac{12}{8} = \boxed{1}\frac{4}{8}$$

$$\frac{4}{8} \times 2 = \frac{8}{8} = \boxed{1}$$

That is, $\frac{3}{8} = (0.011)_2$.

As in the case of base ten not all rationals can be expressed as a finite sum of powers of two. In fact some rationals can have a finite base ten representation but infinite base two:

Example 0.4

Write $\frac{1}{5} = 0.2$ in binary place value representation:

$$\begin{aligned}
\frac{1}{5} \times 2 &= \frac{2}{5} = \boxed{0}\frac{2}{5} \\
\frac{2}{5} \times 2 &= \frac{4}{5} = \boxed{0}\frac{4}{5} \\
\frac{4}{5} \times 2 &= \frac{8}{5} = \boxed{1}\frac{3}{5} \\
\frac{3}{5} \times 2 &= \frac{6}{5} = \boxed{1}\frac{1}{5} \\
\frac{1}{5} \times 2 &= \text{ repeat of line 1}
\end{aligned}$$

That is, $\frac{1}{5} = (0.\dot{0}\dot{0}\dot{1}\dot{1})_2$.

In order to convert an infinite repeating binary decimal like this back to base ten representation we need to find the limit of an infinite series:

$$\begin{aligned}
0.\dot{0}\dot{0}\dot{1}\dot{1} &= 0.001100110011\ldots \\
&= \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^{11}} + \frac{1}{2^{12}} \cdots \\
&= \frac{3}{2^4} + \frac{3}{2^8} + \frac{3}{2^{12}} + \ldots
\end{aligned}$$

The last version of the series above is a geometric series $a + ar + ar^2 + ar^3 + \ldots$ with $a = \frac{3}{2^4}$ and $r = \frac{1}{2^4}$ which therefore has limit

$$\frac{a}{1-r} = \frac{\frac{3}{2^4}}{1 - \frac{1}{2^4}} = \frac{3}{2^4 - 1} = \frac{1}{5}.$$

Example 0.5

Write 0.3 in binary place value representation:

$$\begin{aligned}
0.3 \times 2 &= 0.6 = \boxed{0}.6 \\
0.6 \times 2 &= 1.2 = \boxed{1}.2 \\
0.2 \times 2 &= 0.4 = \boxed{0}.4 \\
0.4 \times 2 &= 0.8 = \boxed{0}.8 \\
0.8 \times 2 &= 1.6 = \boxed{1}.6 \\
0.6 \times 2 &= \text{ repeat of line 2}
\end{aligned}$$

That is, $0.3 = (0.0\dot{1}\dot{0}\dot{0}\dot{1})_2$.
And

$$\begin{gathered}
0.0\dot{1}\dot{0}\dot{0}\dot{1} = 0.0100110011001\ldots \\
= \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{10}} + \frac{1}{2^{13}} + \ldots \\
= \frac{9}{2^5} + \frac{9}{2^9} + \frac{9}{2^{13}} + \ldots
\end{gathered}$$

which is a geometric series with $a = \frac{9}{2^5}$ and $r = \frac{1}{2^4}$ which therefore has limit

$$\frac{\frac{9}{2^5}}{1 - \frac{1}{2^4}} = \frac{9}{2^5 - 2} = \frac{9}{30} = \frac{3}{10} = 0.3$$

0.1.1.1 Finite and infinite decimals.
It is easy to see that any finite decimal can be converted to the form $\frac{a}{b}$ with $a, b \in \mathbb{Z}$.

Example 0.6

$$\begin{aligned}
0.125 &= \frac{125}{10^3} = \frac{1}{8} \\
0.0017 &= \frac{17}{10^4} \\
0.11368 &= \frac{11368}{10^5} = \frac{11421}{125000}
\end{aligned}$$

In general we can write

$$0.d_1 d_2 \ldots d_n = \frac{d_1 d_2 \ldots d_n}{10^n} = \frac{a}{b}$$

where $d_1d_2\ldots d_n$ is an integer whose digits are the digits of the decimal and $\dfrac{a}{b}$ is in lowest form (i.e. a and b have no common factors.)

Now

$$0.d_1d_2\ldots d_n = \frac{d_1d_2\ldots d_n}{10^n} = \frac{a}{b}$$
$$\Rightarrow d_1d_2\ldots d_n \times b = a \times 10^n$$

From the properties of integers we can now make the following deduction:
The denominator b cannot have any prime factors other than 2 or 5
If, for instance, 3 was a factor of b then

$$3 \mid b \Rightarrow 3 \mid d_1d_2\ldots d_n \times b \Rightarrow 3 \mid a \times 10^n.$$

However, 3 is not a factor of $a \times 10^n$ because a does not share any factors with b and the only prime factors of 10^n are 2 and 5 and so 3 cannot be a factor of b

It follows then that a fraction $\dfrac{a}{b}$ in lowest form has an infinite decimal expansion in base ten if b has prime factors other than 2 or 5.
Conversely, since $\dfrac{a}{2^m5^n} = a \times (0.5)^m \times (0.2)^n$ is a finite decimal it follows that $\dfrac{a}{b}$ has a finite decimal expansion if b only has prime factors 2 and 5.

Example 0.7

$\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{5}{7}$ will have infinite base ten expansions.
$\dfrac{1}{2}, \dfrac{2}{20}, \dfrac{5}{50}$ will have finite base ten expansions.

The situation for base two decimals is analogous. If $0.d_1d_2\ldots d_n$ is a base two decimal we still have

$$0.d_1d_2\ldots d_n = \frac{d_1d_2\ldots d_n}{10^n}$$

but this time $d_1d_2\ldots d_n$ is a base two integer and 10^n is 2^n in base ten. And so if $0.d_1d_2\ldots d_n = \dfrac{a}{b}$ where $\dfrac{a}{b}$ is in lowest form then

$$(0.d_1d_2\ldots d_n)_{two} = \frac{(d_1d_2\ldots d_n)_{two}}{(10^n)_{two}} = \frac{a}{b}$$
$$\Rightarrow (d_1d_2\ldots d_n)_{two} \times b = a \times (10^n)_{two} = a \times (2^n)_{ten}$$

Then by the same argument as above we conclude that b cannot have any prime factors other than 2. That is b must be of the form 2^n.

Example 0.8

$\frac{1}{3}, \frac{2}{3}, \frac{5}{7}, \frac{1}{5}, \frac{2}{5}, \frac{1}{10}$ etc. will have infinite base two expansions.
$\frac{1}{2}, \frac{3}{8}, \frac{5}{16}$ etc. will have finite base two expansions.

Note: The algorithms for adding and multiplying binary numberals are essentially the same as for base ten numerals.

Example 0.9

Adding with 'carrying'in base two:

$$\begin{array}{r}
1\ 0\ 1\ 0\ \ 1\ \ 1 \\
1\ 0\ 0\ 0\ \ 0\ \ 1 \\
\underline{1\ 0_1 1_1 1_1 0_1\ 1} \\
1\ 1\ 1\ 1\ 0\ \ 0\ \ 1
\end{array}$$

Multiplying by the 10 or 100 etc. in any base moves the decimal point to the right:

$$10011.1001 \times 10 = 100111.001$$
$$10011.1001 \times 100 = 1001110.01$$

Dividing by the 10 or 100 etc. in any base moves the decimal point to the left:

$$10011.1001 \div 10 = 1001.11001$$
$$10011.1001 \div 100 = 100.111001$$